Douglas's Range Inclusion Theorem

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Overview

The following notions are introduced in Banach / Hilbert space settings.

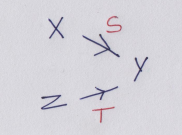
- Range inclusion of operators
- Majorization of operators
- Factorization of operators.

We discuss famous **Douglas Theorem** (sometimes it is called **Douglas's Range Inclusion Theorem**) which tells existence of a close relationship between the notions of range inclusion, majorization and factorization in each of the following operators.

- bounded operators on Hilbert spaces ;
- bounded operators on Banach spaces ;
- unbounded operators on Hilbert spaces.

First Notion : Range Inclusion

Let X, Y, Z be Banach spaces. Let $S \in B(X, Y)$ and $T \in B(Z, Y)$. We have subspaces $R(S)$ and $R(T)$ in Y.



Suppose $R(S) \subseteq R(T)$ or $R(T) \subseteq R(S)$. [**Range Inclusion**]

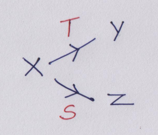
We will find an operation equation involving the operators S and T, using Douglas' Theorem.

Definition 1.

Let X, Y, Z be Banach spaces. Let $T \in B(X, Y)$ and $S \in B(X, Z)$. We say that T **majorizes** *S (or, S is* **majorized by** *T) if there exists $M > 0$ such that*

$$\|Sx\| \leq M \, \|Tx\|$$

for all $x \in X$.



If T majorizes S, then

$$N(T) \subseteq N(S). \tag{1}$$

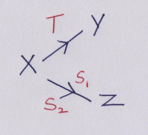
In other words, **majorizing operator has a smaller nullspace.**

We shall prove that converse of (1) is also true when T has a closed range.

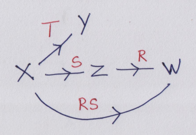
Some Consequences of Majorization

We discuss first some consequences of the property "majorization". Assume that $T \in B(X, Y)$.

If $S_1, S_2 \in B(X, Z)$ and T majorizes S_1 and S_2, then T majorizes $S_1 + S_2$.



If $S \in B(X, Z)$, $R \in B(Z, W)$ and T majorizes S, then T majorizes RS.



Some Consequences of Majorization

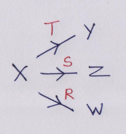
In particular, when all the operators involved are in $B(X)$, then the set of operators majorized by T is a **left ideal** of $B(X)$. That is,

$$\mathcal{A}_T = \Big\{ S : S \text{ is majorized by } T \Big\}$$

is a left ideal of $B(X)$.

Proposition 2.

Let $T \in B(X,Y), S \in B(X,Z)$ and $R \in B(X,W)$. If T majorizes S and S majorizes R, then T majorizes R.



Characterization

Proposition 3.

Assume that $T \in B(X,Y)$ and $S \in B(X,Z)$. The following statements are equivalent:

1. *T majorizes S.*
2. *whenever $\{x_n\} \subseteq X$ with $\|Tx_n\| \to 0$ then $\|Sx_n\| \to 0$.*
3. *there exists $V \in B(\overline{R(T)}, Z)$ such that $S = VT$.*

FA-1(P-76)P-1

Exercise 4.

Let M be a subspace of a normed space X and T be a continuous linear operator from M to a Banach space Y. Prove that T can be extended continuously from M to its closure, $\overline{M}$. Moreover, the extended operator has norm as with the norm of T.

Characterization for Closed Range

For $T \in B(X, Y)$, define $Q_T : X \to X/N(T)$ (called **quotient operator**) by

$$Q_T(x) = x + N(T).$$

Then T has closed range iff there exists $M > 0$ such that

$$\|Q_T(x)\| = \|x + N(T)\| \leq M \|Tx\|, \text{ for all } x \in X.$$

Hence by the terminology of majorization, we have the following characterization for an operator to have closed range.

Proposition 5.

$R(T)$ is closed iff T majorizes Q_T.

Any operator majorizing its quotient operator always has a closed range.

Majorization

We have observed that if T majorizes S, then $N(T) \subseteq N(S)$. The converse is not true, in general.

Example 6.

Consider $X = \ell_2$. Let $T : \ell_2 \to \ell_2$ be defined by

$$T(x_1, x_2, \ldots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$$

and $S = I$, the identity operator. Here $N(T) = N(S) = \{0\}$. Suppose that T majorizes I. Then T has a closed range, a contradiction.

But the converse is true when T has a closed range.

Proposition 7.

Assume that $T \in B(X, Y)$ with $R(T)$ closed. If $S \in B(X, Z)$ has $N(T) \subseteq N(S)$, then T majorizes S. FA-1(P-97)P-5

Majorization

Suppose that T majorizes S.

If an operator (S, or T) has a "property", does the another operator have the same property?

Proposition 8.

Assume that $T \in B(X, Y)$, $S \in B(X, Z)$ and that T majorizes S. That is, $\|Sx\| \leq M \|Tx\|$, for all $x \in X$.

1. *If $R(S)$ is closed and $N(T) = N(S)$, then $R(T)$ is closed.*
2. *If T is compact, then S is also compact.*

Spectral Radius

Let $T \in B(X)$, let

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$$

denote the **spectral radius** of T.

Proposition 9.

Let $T, S \in B(X)$.

If T majorizes S and if $TS = ST$, then T^n majorizes S^n, for $n \geq 1$.

Also $r(S) \leq M\ r(T)$. *FA-1(P-79)P-3*

Spectral Radius

Definition 10.

Let X be a Banach space. An operator $T \in B(X)$ is said to be **quasinilpotent operator** *if $\sigma(T) = \{0\}$.*

Example 11.

Let $H = \ell_2$. Define $T : \ell_2 \to \ell_2$ by

$$T(x_1, x_2, \dots) = \left(0, \frac{x_1}{2}, \frac{x_2}{2^2}, \dots, \frac{x_n}{2^n}, \dots\right).$$

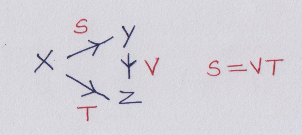
Then T is quasinilpotent. FA-1(P-99)E-6

Proposition 12.

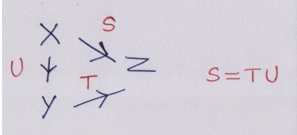
Assume that $T \in B(X, Y)$, $S \in B(X, Z)$ and that T majorizes S. If T is quasinilpotent, then S is also quasinilpotent. FA-1(P-80)P-3a

Assume that $S \in B(X, Y)$ and $T \in B(X, Z)$.

Then S is a **left multiple** of T if there exists $V \in B(Z, Y)$ with $S = VT$.



S is a **right multiple** of T if there exists $U \in B(X, Z)$ with $S = TU$.



That is, if either $S = VT$ or $S = TU$, we say that S **is factored with respect to** T. If S and T have a **common domain**, we get a left multiple of T for S. If S and T have a **common co-domain**, we get a right multiple of T for S.

Douglas Theorem for Bounded Operators on Hilbert Spaces

We now discuss existence of a close relationship between the notions of majorization, factorization and range inclusion for operators on a Hilbert space.

Douglas[1] discovered these relations in the study of an unpublished manuscript of deBranges and Rovnyak.

[1] Douglas, R. G, On majorization, factorization, and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.* **17** (1966), 413-415.

Douglas Theorem for Bounded Operators on Hilbert Spaces

In the following result, **we shall see the equivalence of three notions in the order : range inclusion, majorization and factorization.**

Theorem 13 (Douglas, 1966).

Let A and B be bounded operators on a Hilbert space H. The following statements are equivalent:

1. $R(A) \subseteq R(B)$.
2. *There exists $M > 0$ such that $\|A^* x\| \leq M \; \|B^* x\|$ for all $x \in H$.* *That is, there exists $M > 0$ such that $AA^* \leq M^2 \; BB^*$.*
3. *There exists a bounded operator C on H so that $A = BC$.* *FA-1(P-93)T-1*

The plan of the proof is given below :

$$(1) \implies (3) \implies (1) \qquad \text{and} \qquad (2) \implies (3) \implies (2).$$

Closed Graph Theorem

To prove the "Douglas Theorem" for Bounded Operators on Hilbert Spaces, we use the closed graph theorem.

Theorem 14 (Closed Graph Theorem).

Let X and Y be Banach spaces and $T : X \to Y$ be linear. Then T is bounded iff T is closed (the graph of T is closed).

Easy Generalization

If we consider operators A and B with domains equal to the Hilbert spaces H_1 and H_2, respectively, but having common range H, then we need only to modify the statement of the theorem so that the operator C is now defined from H_1 to H_2 to obtain parallel result for this case. **The proof is exactly the same.**

Theorem 15.

Let H_1, H_2, H be Hilbert spaces. Let $A \in B(H_1, H)$ and $B \in B(H_2, H)$. The following statements are equivalent:

1. $R(A) \subseteq R(B)$.
2. *There exists $M > 0$ such that $\|A^*x\| \leq M \, \|B^*x\|$ for all $x \in H$. That is, there exists $M > 0$ such that $AA^* \leq M^2 \; BB^*$.*
3. *There exists $C \in B(H_1, H_2)$ so that $A = BC$.*

Douglas Theorem for Bounded Operators on Hilbert Spaces

We proved the following Douglas' Theorem for bounded operators on Hilbert spaces.

Theorem 16 (Douglas, 1966).

Let A and B be bounded operators on a Hilbert space H. The following statements are equivalent:

1. $R(A) \subseteq R(B)$.
2. *There exists $M > 0$ such that $\|A^*x\| \leq M\ \|B^*x\|$ for all $x \in H$. That is, there exists $M > 0$ such that $AA^* \leq M^2\ BB^*$.*
3. *There exists a bounded operator C on H so that $A = BC$.*

Summary : Douglas Theorem for Bounded Operators on Hilbert Spaces

We know that every Hilbert space operator A has an adjoint (denoted by A^*) and $A = A^{**}$.

Let A and B be bounded operators on a Hilbert space H. The following statements are equivalent:

1. $R(A)$ is smaller than $R(B)$.
2. A^* is majorized by B^*.
3. If A **is a right multiple of** B (B is on the left side, say, $A = BC$, for some C).

1. $R(A^*)$ is smaller than $R(B^*)$.
2. A is majorized by B.
3. If A **is a left multiple of** B (B is on the right side, say, $A = CB$, for some C).

The case in which the spaces X, Y, Z are Banach

An obvious question to ask is whether the Douglas theorem generalizes to the case in which the spaces are Banach.

Let X, Y, Z **be Banach spaces.** Let $A \in B(X, Y)$ and $B \in B(Z, Y)$.

The condition for majorization

$$\|A^* x\| \leq M \, \|B^* x\|$$

can be interpreted as

$$\|A^* y^*\| \leq M \, \|B^* y^*\|$$

and all $y^* \in Y^*$.

Here Y^* is the **dual space** of Y.

Notations

1. The **dual space** of X is denoted by X^*.
2. For $z \in Z$ and $z^* \in Z^*$, we denote $z^*(z)$ for $\langle z, z^* \rangle$.
3. For $S \in B(X, Y)$, $S^* \in B(Y^*, X^*)$ is the **usual adjoint** of S. Hence,

   for $x \in X, y^* \in Y^*$, we write

   $$\langle Sx, y^* \rangle = \langle x, S^* y^* \rangle$$

   instead of $y^*(Sx) = S^* y^*(x)$.
4. $N^\perp := \left\{ x^* \in X^* : x^*(s) = 0, \forall s \in N \right\}$, for any $N \subseteq X$.
5. $M_\perp := \left\{ x \in X : m(x) = 0, \forall m \in M \right\}$, for any $M \subseteq X^*$.

   One can prove that if $N_1 \subseteq N_2 \subseteq X$, then $N_1^\perp \supseteq N_2^\perp$.
6. $R(A)^\perp = N(A^*)$.

Douglas Theorem for Bounded Operators on Banach Spaces

In order to extend Douglas factorization theorem to Banach spaces, generally speaking, one needs to consider the range inclusion of the adjoint operators instead, rather than the operators themselves.

This was done by Embry[2].

[2] Mary R. Embry, Factorization of operators on Banach space, *Proc. Amer. Math. Soc.* **38**(2) (1973), 587-590.

Douglas Theorem for Bounded Operators on Banach Spaces

We now prove that the Douglas theorem remains valid for adjoints of operators on Banach spaces.

Theorem 17 (Embry, 1973).

Let $\mathbb{A}$ and $\mathbb{B}$ be bounded operators on a Banach space X. The following statements are equivalent:

1. $R(\mathbb{A}^*) \subseteq R(\mathbb{B}^*)$.
2. $\|\mathbb{A}x\| \leq M \, \|\mathbb{B}x\|$ *for some* $M > 0$ *and all* $x \in X$.
3. $\mathbb{A} = C\mathbb{B}$ *for some bounded operator* $C : \overline{R(\mathbb{B})} \to X$.

Douglas Theorem for Bounded Operators on Banach Spaces

The plan of the proof is given below :

$$(2) \implies (3) \implies (1) \implies (3) \implies (2).$$

Let us note first that this theorem indeed generalizes Douglas theorem. To see this, let $A = \mathbb{A}^*$ and $B = \mathbb{B}^*$, which is possible since every Hilbert space operator has an adjoint. Then the second and third statements in the two theorems are identical.

The condition

$$\mathbb{A} = C\mathbb{B} \text{ for some bounded operator } C : \overline{R(\mathbb{B})} \to X.$$

becomes

$$A^* = CB^* \text{ for some bounded operator } C : \overline{R(B^*)} \to X.$$

But since X is a Hilbert space, C has a continuous linear extension G on X so that

$$A^* = GB^*.$$

Thus

$$A = BG^*$$

which retrieves the first statement of Douglas theorem for bounded operators on Hilbert spaces.

Does Douglas Theorem hold true on Banach Spaces ?

Let X, Y, Z be Banach spaces. Let $A \in B(X, Y)$ and $B \in B(Z, Y)$.

Let us discuss all possible relations among the following:

1. $R(A) \subseteq R(B)$.
2. There exists $M > 0$ such that $\|A^* y^*\| \leq M \|B^* y^*\|$ for all $y^* \in Y^*$.
3. There exists a bounded operator $C \in B(X, Z)$ so that $A = BC$.

"(3) $\implies$ (1)" and "(3) $\implies$ (2)" are obvious.

Proposition 18.

"$R(A) \subseteq R(B)$" $\implies$ "A^* majorized by B^*" is also always true. FA-1(P-80)P-4

To prove "$R(A) \subseteq R(B)$" $\implies$ "A^* majorized by B^*", we use the following well-known fundamental result.

Theorem 19 (Uniform Boundedness Principle).

Suppose X is Banach, Y is a normed space and $\mathcal{A} \subseteq \mathcal{B}(X, Y)$. If $\mathcal{A}$ is pointwise bounded, then $\mathcal{A}$ is uniformly bounded.

"$R(A) \subseteq R(B)$" $\implies$ "$A = BC$ (for some C)" is not true, in general.

We shall see an example, given by Douglas.

Proposition 20.

If $N(B)$ is complemented, then "$R(A) \subseteq R(B)$" $\implies$ "$A = BC$ (for some C)".

FA-1(P-86)

Douglas' Example

Douglas (unpublished manuscript) gave a counter example of bounded operators A and B for which $R(A) \subseteq R(B)$ is true but there is no operator C such that $A = BC$. The example appeared in a paper of Embry M.R. [1973].

Example 21.

Let X be a Banach space, N a subspace of X, and Y the set of bounded functions on the integers so that

$$f(n) = \begin{cases} X & \text{when} \quad n \leq 0 \\ X/N & \text{when} \quad n > 0 \end{cases}$$

Y is a Banach space with respect to

$$\|f\| = \sup \|f(n)\|.$$

FA-1(P-82)

Douglas' Example (contd...)

Consider the operators A and B on Y defined by

$$(Af)(n) = \begin{cases} f(n) & \text{for } \; n = 1 \\ 0 & \text{for } \; n \neq 1 \end{cases} \qquad \text{and} \qquad (Bf)(n) = \begin{cases} \pi f(0) & \text{for } \; n = 1 \\ f(n-1) & \text{for } \; n \neq 1 \end{cases},$$

where π is the natural map from X to X/N.

Then $R(A) \subseteq R(B)$.

Suppose that there exists an operator C on Y such that

$$A = BC.$$

Douglas' Example (contd...)

Let D_1 be the map from X/N to Y and D_2 the map from Y to X defined by

$$(D_1(x+N))(n) = \begin{cases} x+N & \text{for } \; n=1 \\ 0 & \text{for } \; n \neq 1 \end{cases} \quad \text{and} \qquad (D_2 f)(n) = f(0).$$

Hence $E = D_2 C D_1$ is a map from X/N to X such that $\mathtt{I} - E\pi$ is a bounded projection of X onto N.

Thus if we choose N to be a subspace for which no bounded projection exists, then we arrive at a contradiction and see that there exists no operator C on Y for which $A = BC$.

In the example, **we have seen that there are operators A and B for which $R(A) \subseteq R(B)$ is true but there is no operator C such that $A = BC$.**

"A^* is majorized by B^*" $\implies$ "$R(A) \subseteq R(B)$" is not true, in general.

Bouldin[3] has given a counter example of bounded operators on a Banach space for which "B^* majorizes A^*" is true but "$R(A) \subseteq R(B)$" is false.

Example 22.

Define A and B on c_0 by $\quad Ae_k = 0 \qquad$ *for* $k \neq 1$

$$Ae_1 = y = \left(\frac{1}{2}, \frac{1}{2^2}, \dots\right)$$

and

$$B(x_1, x_2, \dots) = \left(\frac{x_1}{2}, \frac{x_2}{2^2}, \dots\right).$$

We have $(c_0^*)^* = \ell_1^* = \ell_\infty$ *and* $R(A^{**}) \subseteq R(B^{**})$.

By Theorem (17), A^* *is majorized by* B^**, That is, there exists some* $M > 0$ *such that* $\|A^*f\| \leq M\|B^*f\|$ *for all* $f \in \ell_1$*. But* $R(A) \nsubseteq R(B)$*.* FA-1(P-87)

[3] Richard Bouldin, A counterexample in the factorization of Banach space operators, *Proc. Amer. Math. Soc.* **68** (3) (1978), 327.

Proposition 23.

If Z is reflexive, then "A^* majorized by B^*" $\implies$ "$R(A) \subseteq R(B)$". *FA-1(P-89)*

"A^* majorized by B^*" $\implies$ "$A = BC$ (for some C)" is not true, in general.

Proposition 24.

If Z is reflexive and $N(T)$ is complemented in Z, then "A^* majorized by B^*" $\implies$ "$A = BC$ (for some C)".

FA-1(P-91)

Douglas Theorem for Unbounded Operators on Hilbert Spaces

Furthermore, Douglas extended the above result to the case of unbounded operators on Hilbert spaces as follows.

Theorem 25.

Let A and B be closed densely defined operators on a Hilbert space H.

1. *If $AA^* \leq BB^*$, there exists a contraction C so that $A \subset BC$.*
   (The statement $AA^* \leq BB^*$ is assumed to mean that $D_{BB^*} \subset D_{AA^*}$ and for $x \in D_{BB^*}$ we have $\langle AA^*x, x\rangle \leq \langle BB^*x, x\rangle$.)
2. *If C is an operator so that $A \subset BC$, then $R(A) \subset R(B)$.*
3. *If $R(A) \subset R(B)$, then there exists a densely defined operator C so that $A = BC$ and a number $M > 0$ so that*

$$\|Cx\|^2 \leq M\{\|x\|^2 + \|A^*x\|^2\} \quad \text{for } x \in D_C.$$

Moreover, if A is bounded, then C is bounded and if B is bounded, then C is bounded.

Consequences of Douglas Theorem

- Several results on operators with closed range.
- Existence of Moore-Penrose inverses of operators.
- Rich theory on quotient of operators.
- Operator theoretic approach in inequalities.
- Study of perturbations of an operator by compact operators.

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